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The Peculiar Properties of Quantum Hall Systems

Bertrand Halperin

Abstract

For more than two and a half decades, experiments on two dimensional electron systems in strong magnetic fields, at low temperatures have given rise to a series of surprising phenomena, known collectively as the quantum Hall effects. I shall give here a brief overview of the field, beginning with a description of the integer and fractional quantized Hall effects, and the basic conditions associated with their occurrence: an absence of mobile carriers in the bulk of the 2D system, combined with unidirectional electrical conduction along the sample boundary. The way in which these conditions are satisfied, in the case of the integer quantized Hall effect can be understood in terms of non-interacting electrons, using a semiclassical argument. The fractional quantized Hall effect depends fundamentally on electron-electron interactions, which lead to a peculiar highly correlated ground state, with elementary quasiparticle excitations that carry fractional charge and "fractional statistics." I shall discuss briefly a number of other phenom-
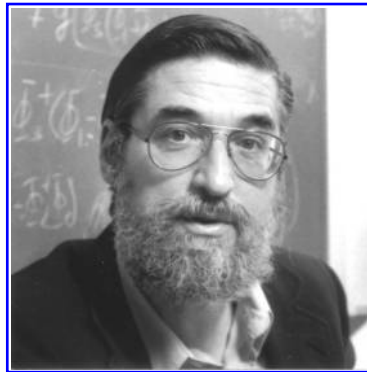
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ena that occur in quantum Hall systems, and indicate some areas where open questions exist.

Introduction

The term "quantum Hall effects" refers to a large set of peculiar phenomena, which occur in two-dimensional electron systems, at low temperatures in strong magnetic fields.[1–8]

In experimental realizations, the two-dimensional system has usually been formed from electrons (or holes) in a gallium arsenide semiconductor structure. Very high quality samples of this type can be grown by the process of molecular beam epitaxy. The earliest studies of quantized Hall effect, however, were done in silicon field-effect transistor structures (MOSFETS). Most recently, the integer quantized Hall effect has been seen in graphene, a single atomic layer of graphite, and also in double-layer graphene. A large variety of phenomena occur, in part, because samples can differ widely in electron densities and in the degree of freedom from defects, and the magnetic fields applied can vary over a large range, from approximately 0.1 to 45 Tesla. Still, it is quite remarkable what a large variety of phenomena have been found in a restricted class of devices, made from relatively simple basic materials.

Experiments on quantum Hall systems have, in fact, produced many surprises since the discovery of the Integer Quantized Hall Effect, in 1980. Understanding these results has required concepts and mathematical techniques from far corners of theoretical physics, including some completely new ideas. The goal of this talk will be to give a brief overview of the subject. The questions I will try to address, at least briefly, are:

> What are the quantized Hall effects?
> How do we understand them?
> What are some unanswered questions in the field?

A typical geometry for studying the Hall effect in a two-dimensional system is shown in Figure 1. An electrical current I_x is passed through the sample, in the x-direction, through current contacts at the two ends. Voltage probes, which draw no current, are attached at several points along the lateral edges, and one measures simultaneously the longitudinal voltage drop V_x between two probes on the same edge, and the "Hall voltage", V_y between contacts lined up on opposite edges of the strip. As the voltages are generally proportional to the current, it is convenient to divide the voltages by the
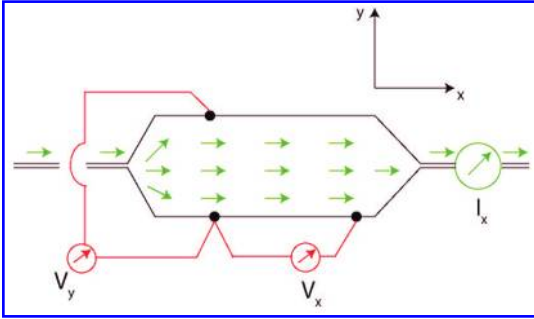
Figure 1: Sketch of arrangement for measurement of the longitudinal resistance and Hall resistance in a two-dimensional electron system. A current I_x flows along the sample in the x-direction, entering and leaving through the current contacts at the two ends. Voltage is measured at contacts along the edges, which draw no current. Hall voltage V_y is the voltage difference between two contacts on opposite edges of the sample, while longitudinal voltage V_y is the voltage drop between two contacts on the same edge. A magnetic field B is applied perpendicular to the plane of the sample.

current, and discuss the longitudinal resistance, $R_{xx} \equiv V_x/I_x$, and the Hall resistance, $R_H \equiv V_y/I_x$

The *classical* Hall effect was discovered in 1879 by Edwin Hall, who observed the effect in three-dimensional metals, and formulated a classical theory for it. In the classical Hall effect, the Hall voltage V_y is proportional to the magnetic field B. More precisely, the Hall resistance is given by $R_H = B/n_e e$, where n_e is the density of electrons, and e is the electron charge. The longitudinal resistance will depend on details, including the shape of the sample and the amount of electron scattering in the material, but it is roughly independent of B in most cases.

Quantum mechanics predicts corrections to the classical Hall effect, which become manifest particularly at low temperatures, in strong magnetic fields, in high quality samples, which have a long mean-free-path for electrons between scattering events due to impurities. These corrections include oscillations in R_{xx} and R_H as a function of the applied magnetic field, which had been known for many years in three-dimensional systems, and were studied in two dimensional electron systems during the 1970s. Nonetheless, the discovery, in 1980, by von Klitzing, Dorda and Pepper, [9] of what we now know as the Integer Quantized Hall effect, was a very great surprise. In measurements on a silicon MOSFET, as they varied the magnetic field, they found a series of plateaus, where the Hall resistance was absolutely constant over a certain interval of B. In these same field intervals,

the longitudinal resistance was found to become vanishingly small at low temperatures. Most remarkably, the values of R_H on the various plateaus could be fit by a simple formula:

$$R_H = h/\nu e^2, \tag{1}$$

where h is Planck's constant, and ν is an integer that varies from one plateau to the next. (In common units, the quantity h/e^2 is equal to 25,812.82 ohms.) Subsequent experiments, in many laboratories, found the same behavior in a variety of semiconductor materials, and have confirmed that the Hall plateaus are quantized in integer ratios within a precision of better than a part in 10^7. It should be emphasized that the observed values of the quantized Hall plateaus do not depend on the shape of the sample, nor on other details of the preparation, within broad limits.

Following the discovery of the integer quantized Hall effect, a number of theoretical arguments were advanced to explain the precision of the quantization, despite the presence of impurities and irregularities in any real sample. Among the explanations were elegant arguments using the concept of gauge invariance or considerations of the topological invariants of an electron wave function in a magnetic field. [10,11] There were also arguments based on perturbation theory, including some in the earlier literature that, in retrospect, had pointed strongly in the right direction. [12] I will not review these arguments here, but in my discussions below, I will present a somewhat different approach to understanding the precision of the quantized Hall plateaus, emphasizing the role of sample edges. [13]

Even more surprising than the integer quantized Hall effect was the discovery, in 1982, of the fractional quantized Hall effect. [14] In samples of very high quality, in addition to the integer quantized Hall plateaus, at sufficiently high magnetic fields, one may observe additional plateaus, where ν is a simple rational fraction. The first fractions seen were at $\nu = 1/3$ and 2/3. Shortly thereafter, with the advent of higher quality samples, stronger magnetic fields, and lower measurement temperatures, Hall plateaus were seen (or at least indicative dips in the longitudinal resistance) at a large number of *odd-denominator* fractions, including $\nu = 4/3, 5/3, 1/5, 2/5, 3/5, 3/7, 4/7, 4/9$, and 5/9.[1]

Although explanations of the integer quantized Hall plateaus need to take into account the effects of impurities in a fundamental way, the effects of electron-electron interactions are generally secondary. Essentially, the integer quantized Hall effect would be present in a system of non-interacting electrons, and from a theorists point of view, it is only necessary to show that

the effect would not spoiled by interactions between electrons, at least if the interactions are not too strong. The fractional quantized Hall plateaus, however, could not be even remotely explained in terms of non-interacting electrons. The role of electron-electron interactions is absolutely essential, and we now understand that the explanation involves new types of highly correlated electron states, which had never been previously encountered.

What about *even-denominator* fractions? Quantized Hall plateaus have been found corresponding to a few even denominator fractions; e. g., $\nu = 5/2$. However, there is *no* quantized Hall plateau in a single layer sample at the simplest even-denominator fraction, $\nu = 1/2$. In the magnetic field region where the Hall resistance passes through the corresponding value of $2h/e^2$, the Hall resistance varies linearly with magnetic field, just as in the classical Hall effect. The longitudinal resistance varies only slowly with magnetic field in this region. However, strong anomalies have been observed in other properties, such as in the propagation velocity of surface acoustic waves. [15] Thus, there is still something very strange occurring at the point where the Hall resistance passes through the value corresponding to $\nu = \frac{1}{2}$. I like to call this the "Unquantized Quantum Hall Effect". The explanation for the observed behavior is, in fact, a very interesting story, but, unfortunately, I will not be able to discuss it in this talk. [16, 17]

Conditions for the quantized Hall effect, integer or fractional

In order to have a quantized Hall plateau, the electron system should be in a state that has several special characteristics. In the 2D bulk, far from the edges, the system should have an energy gap for creation of mobile charges. Then, at low temperatures, the charge carriers will freeze out, so the bulk is essentially an insulator.

The energy gap must vanish along the edges of the sample, so that electrical current can flow along the edges. In fact, the edge of quantized Hall system is a peculiar type of one-dimensional conductor, referred to as a "chiral metal": charge carriers travel in only one direction along each edge.

The electrical current along a chiral edge is determined by the voltage V on the edge. For a small change in voltage, δV, change in current will be given by

$$\delta I = G \, \delta V, \tag{2}$$

where G is a constant. The value of G determines the quantized Hall resistance, as we shall shortly see.
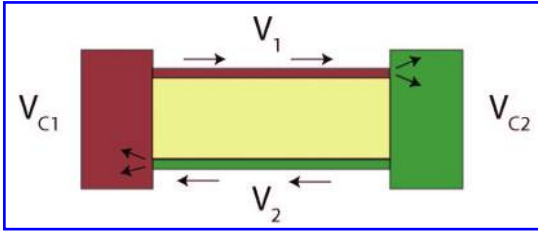
Figure 2: Situation when sample is in a quantized Hall state (integer or fractional). The bulk of the sample is an insulator, which carries no transport current. Sample edges are "chiral conductors", which can carry current in only one direction (indicated by arrows), proportional to the voltage on the edge. The voltages V_1 and V_2 are constant on each edge, and in the case of ideal contacts, are equal to voltage (V_{C1} or V_{C2}) of the contact from which the edge current flows.

Figure 2 illustrates the current flow around a quantized Hall strip with current contacts at the ends. If the two end contacts have the same voltage, $V_{C1} = V_{C2}$, then the system will be in equilibrium, with the voltage everywhere equal to the voltage of the end contacts. In equilibrium, there should then be no net current I flowing from left to right. In order for this to be true, if Eq. (2) applies, it must be true that the conductance G is the same on the top and bottom edges, so $I = I_1 - I_2 = 0$. Also, because electrical charge is a conserved quantity, it follows that if the bulk of the 2D system is an insulator, so there is no scattering of charges between the edges, then the current on each edge must be a constant, independent of position along the edge. Hence G must be the same everywhere along the two edges.

If the two end contacts have different voltages, then the voltages on the two edges will be different from each other, $V_1 \neq V_2$, and there will then be a net current (Hall current) flowing along the sample. Since $\delta I_1 = G \delta V_1$, and $\delta I_2 = G \delta V_2$, we now have:

$$I = I_1 - I_2 = G(V_1 - V_2). \tag{3}$$

If there is no scattering of charges between the edges, the current on each edge must still be constant along the edge by charge conservation. Hence V_1 and V_2 are constants along each edge. *There is no voltage drop along a given edge.* If we attach voltage contacts as in Figure 1, we will find that $R_{xx} = 0$, while $R_H = 1/G$.

In the case of ideal current contacts, the voltage on each edge of the quantized Hall system will be identical to the voltage of the end contact

from which the current flows into that edge. For non-ideal contacts, there may be a small voltage difference between the current contact and the edge, but the voltage will still be a constant along each edge.

We have seen that if there are no mobile carriers in the bulk of the sample, then G must be precisely constant along any edge. Therefore, *this value must be independent of any details of the edge or of the sample that could vary from one place to another.* Consequently, one must be able to compute the value of G in a simple model with ideal edges and with no impurities or disorder. In fact, the only way that G can be precisely a constant along an edge is if it is quantized, i. e., it must be restricted to a discrete set of values

We may note that a *small* concentration of impurities will not affect the Hall quantization, as it will produce only a small number of localized electron states in the bulk, which cannot carry current across the sample, from one edge to the other. The quantized Hall effect will only break down if concentration of impurities becomes high enough so that the impurity states overlap strongly and produce a new path for electrons to cross the sample.

Example: The Integer QHE

Consider non-interacting electrons in uniform magnetic field B in 2D. In *classical mechanics*, one finds that electrons move in circles, at a cyclotron frequency ν_C, proportional to B. The radius of the circle depends on the energy of the particle, which is arbitrary, and location of the center of the circle is also arbitrary, but the cyclotron frequency is fixed. In *quantum mechanics*, however, the energy levels for orbital motion are quantized into discrete levels, known as "Landau levels", with energies given by

$$E_n = h\,\nu_C(n1/2), \quad n = 0, 1, 2, 3, \dots \tag{3}$$

(These energy levels are the same as one would have for a simple harmonic oscillator with frequency ν_C.) The number of independent orbits in each Landau level (i. e., the number of independent positions for the centers of the circles) is also quantized, and is given by

$$N_B = B\,e\,A/h, \tag{4}$$

where A is the area of the sample. (When the electron spin is taken into account we find that each orbital level is split into two states, whose energies

differ by the Zeeman energy, arising from the coupling of the spin magnetic moment to the applied magnetic field.) In any case, we may define a very important dimensionless parameter, the *Landau level filling factor f* , by

$$f \equiv N_e/N_B, \tag{5}$$

where N_e is the number of electrons in the system. The filling factor f may be alternately described as the number of electrons per quantum of magnetic flux.

We shall see that the integer QHE occurs, typically, when f is an integer, and the value of the quantized Hall conductance $(1/R_H)$ is, here, $G = fe^2/h$.

Landau levels in a strip of finite width

Let us now consider the situation in a strip of large but finite width. Suppose the electrons are confined by potential walls to a strip, $0 < y < W$, by a potential, which is strongly repulsive at the boundaries but zero in the middle of the strip. The electron energy levels will then be pushed up by the effects of the potential when the centers of the orbits get close to the edge of the strip. The overall situation may be understood with the sketch in Figure 3, showing the energies of the allowed orbits, versus the position y at the center of the orbit. (For simplicity, we neglect the electron spin in this figure.) Far from the edges, the energy levels are given by the values in Equation [3], and are independent of position. However, for each value of the Landau level index n, the energy increases in the vicinity of the edges, eventually rising without limit, if the confining potential is infinitely high.

For non-interacting electrons, in the ground state, at zero temperature, the energy levels are filled up to an energy E_F, known as the Fermi level. In the figure, we have chosen the Fermi level so that it lies between the second and third energy level in the bulk. Thus there are precisely two filled Landau levels in the bulk, so that $f = 2$, and there is an energy gap in bulk. At each edge, however, the lowest two Landau levels are pushed up through the Fermi energy, so there are two conducting states at the Fermi level, with no energy gap separating the filled and empty levels. It can be shown that electrons in these edge states travel along the edge, in opposite directions at the two edges. Each edge state contributes an amount e^2/h to G, so the total Hall conductance is, here, $G = 2e^2/h$.
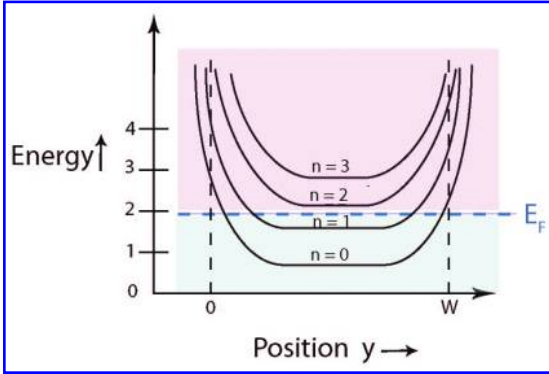
Figure 3: Energy levels for (spinless) non-interacting electrons in a magnetic field, confined to a strip of finite width W. Vertical axis is energy; horizontal axis indicates position y across the strip. Energy levels are discrete and independent of position far from the edges, but are pushed upwards in the vicinity of the edge. Electron levels are filled up to the Fermi energy, E_F.

Fractional quantized Hall states

For the fractional quantized Hall effect (FQHE), the Fermi energy in the bulk is in the middle of a partially filled Landau level. Why should there be an energy gap in this case? As mentioned in the Introduction, the energy gap in this case must come entirely from electron-electron interactions. However, it is highly non-trivial to understand where this might come from. In normal metals, if electron-electron interactions are not too strong, their effects can be understood by the well-established mathematical method known as "perturbation theory". However, in order to use perturbation theory, it is necessary to have a unique ground state for the non-interacting problem to start from, and not too many nearby states with very low total energy. For the integer quantized Hall effect, this is not a problem because there is a unique ground state and excited states in the bulk are separated from it by an energy gap. For a partially filled Landau level, however, there are a very large number of ways of placing electrons in orbits within the Landau level, and all these states have identical energies, if the electron-electron interaction is absent

A big breakthrough in solving this problem was made in 1983 by Laughlin, [18] who proposed a unique ground state for the interacting electron system at $f = 1/3$, and several other odd-denominator filling fractions. Subsequent work, by many authors, have extended Laughlin's work, and have introduced alternative theoretical approaches to understand the many

filling fractions that have been observed. [1–8] All of these approaches involve rather sophisticated mathematics, and I will not try to explain them here. However, I would like to emphasize some of the conclusions.

The ground states for fractional quantized Hall systems states are strongly-correlated many-body states with very peculiar properties. One peculiarity is that the elementary charged excitations are quasiparticles with *fractional charge*. [18] For example, the FQHE state at $\nu = 1/3$ has quasiparticles with charge $= \pm e/3$; the FQHE state at $\nu = 2/5$ has quasiparticles with charge $= \pm e/5$; etc. This means that an electron added to the bulk of an FQHE state at $f = 1/3$ can lower its energy by breaking up into three quasiparticles of charge $e/3$. The $e/3$ charges are achieved by small displacements of the surrounding electrons in the FQHE state, moving a net charge of 2/3 away from the immediate vicinity of the added electron and depositing this charge in two nearby places, with 1/3 net charge in each.

Quasiparticles in odd-denominator FQHE states also exhibit a phenomenon known as *fractional statistics* [19,20]. This means that the quantum properties are, in some sense, intermediate between the properties of the two types of particles that occur in normal three dimensional systems, fermions and bosons, (Fermions are particles such as electrons, protons, and neutrons, which obey the Pauli exclusion principle in filling their energy levels, while bosons do not.) However, as was first pointed out by Leinaas and Myrheim in 1977 [21], the mathematical structure of quantum mechanics in two-dimensions does not by itself rule out the possibility of other types of particles. In particular, one could have particles with fractional statistics, the quantum mechanical wave function may be multiplied by a complex phase factor when two identical quasiparticles are interchanged, whereas for fermions or bosons, the wave function can only be multiplied by ± 1.

It was not clear, however, whether such strange particles would ever occur in nature, where they would have to be constructed, somehow, out of ordinary electrons and nuclei, which are fermions or bosons. An analysis of the consequences of Laughlin's wave functions for the ground state and quasiparticles in the $f = 1/3$ state, and related wave functions for other fractional states, showed that, remarkably, the quasiparticles were realizations of the fractional statistics concept. [19,20]

Fractional statistics have consequences that are important, indirectly, for understanding the construction of higher level fractional quantized Hall states, i. e., states at fractions that cannot be described directly by Laughlin's original construction. [19] However, it is more difficult to see effects of fractional statistics directly, particularly in systems with just a few quasiparticles present. One place where fractional statistics should be directly

manifest are in "quantum interference" experiments. [22] Such experiments can be carried out, at least in principle, by constructing a quantized Hall system with several constrictions that are narrow enough for electrons or quasiparticles to tunnel across the constrictions, from one chiral edge to the other. Tunneling of this type leads to breakdown of the quantized Hall plateaus, giving a deviation from the quantized Hall resistance, and non-zero values of the longitudinal resistance R_{xx}. If two constrictions are close together, the value of the measured resistance may exhibit small oscillations as a function of the value of the magnetic field or of the total charge enclosed in the area between the two constrictions, which can be interpreted as a form of quantum interference between portions of the wave function that tunnel through the different constrictions. If there are quasiparticles with fractional statistics enclosed in this area, the extra phase factors arising from the fractional statistics will lead to changes in the pattern of resistance oscillations, which should be observable in experiments. [22]

Very recently seen, such effects have been seen, for $\nu = 1/3$, in experiments by V. Goldman and collaborators. [23] However, there remain many other experiments in similar geometries, with several constrictions, that are still poorly understood. Real systems are indeed complicated, and it is known that in some cases at least, competing tunneling processes may give effects that mask the simple interference effect one is looking for, and can change the period of observed resistance oscillations, even for integer quantized Hall systems. [24] This entire area remains one of active current research.

Even-denominator FQHE at $\nu = 5/2$

In the model proposed by Moore and Read, in 1990, for the even-denominator fractional quantized Hall state at $\nu = 5/2$, the quasiparticles are even more peculiar than in the common odd-denominator FQHE states. [25] In their model, the quasiparticles have charge $\pm e/4$ and obey *non-abelian* statistics. This means that when multiple quasiparticles are present, there are many independent wave functions that can describe the ground state of the system. If multiple quasiparticles are interchanged, the final wave function can be quite different from the wavefunction one started with (not just multiplied by a phase factor) and the state depends on the order in which the quasiparticles are interchanged. Although there are a number of theoretical arguments and numerical calculations that give support to the Moore-Read proposal, there is, so far, no direct experimental evidence that it

is a correct model for the 5/2 state. Recent improvements in sample quality and in experimental techniques give hope, however, that this question may be resolved in the near future. Among the proposed experiments to look for non-abelian statistics are interference experiments, which should have a distinctive signature if the hypothesis is correct. [26]

Interest in non-abelian statistics has been given a boost in recent years by the proposal that such quasiparticles could be used for building a "quantum computer" [26], which could be used to solve certain types of numerical problems that would be impractical for any conventional digital computer. However, we are still a long way from realizing any type of quantum computer, and it is far from clear whether a computer using fractional statistics quasiparticles will ever be feasible in practice.

Other peculiar phenomena in quantum Hall systems

I should like to close by mentioning, at least briefly, a few of the other remarkable phenomena that have been observed in quantum Hall systems, beyond the integer and fractional quantized Hall effects, and the unquantized quantum Hall effect mentioned above.

So far, we have ignored the spin degree of freedom for the electrons. This is generally correct for even-integer quantized Hall states, and for odd-integer Quantum Hall States, this is also correct if the electrons in the highest occupied Landau level are completely spin-aligned by the applied magnetic field. For fractional quantized Hall states, it is also correct to ignore the spin degree of freedom if the electrons in the partially full Landau level are maximally spin-polarized. However there are situations where this is not correct. In the case of electrons in GaAs structures, the effective magnetic moment of the electron is particularly small, about five times smaller than the moment of free electrons, so the Zeeman energy, which favors alignment of the electron spins, can be quite small. Effects of the electron-electron interaction may favor or disfavor alignment of the electron spins, depending on the filling factor f. In some cases, interaction effects may be strong enough to favor ground states or quasiparticle states where the electron spins are not fully aligned [27]. Experimentally, in some cases one has observed phase transitions between fractional quantized Hall states where the spin is fully aligned and states where it is not, as one varies an experimental parameter, such as electron density or a component of applied magnetic field parallel to the plane of the sample. [28–30] Such transitions lead to anomalies in the electrical resistance, which have been studied with

great interest. Deviations from perfect spin alignment due to quasiparticles with multiple reversed spins can also produce dramatic effects at filling fractions close to, but slightly away from, the integer quantized Hall state at $f = 1$. [31]

Electron spins couple to the nuclear spins of Ga and as nuclei via the hyperfine interaction. Flowing an electric current thorough an elctron system with inhomogeneous electron spin polarization can lead to non-equilibrium polarization of the nuclei, which can feed back in turn to the electron system and cause changes in the electrical resistance. Changes in the state of electron polarization can also produce big changes in the relaxation rate for nuclear spins. These effects have been studied in a number of quantized Hall systems, where they can produce quite dramatic effects. [32]

Systems composed of two closely spaced electron layers, separated by a thin barrier, can also lead to a number of unique phenomena in the quantum Hall regime. Modern technology has allowed experimenters to attach separate leads to the layers, giving separate control over the current flow and separate measurements of the voltage drops in the two layers. [33] A fascinating variety of phases can exist in bilayer systems. [34] As one example, in the regime where the total filling factor in the combined layers is one electron per flux quantum ($f = 1$), if the layers are sufficiently close and the electron density is sufficiently small, the Coulomb interaction between electrons in different layers may be comparable to the interaction between neighboring electrons in a single layer. In this case, one has observed a highly correlated phase where the combined system behaves like a quantized Hall conductor for current that moves in the same direction in the two layers, but is more like a superconductor, with neither longitudinal resistance nor Hall resistance, when the current flow is equal and opposite in the two layers. [33] If there is a current flow in one layer and not the other, one observes a quantized Hall voltage which is the same in the layer without current as it is in the layer with current. (This has been called quantized Hall drag.)

There are a variety of experiments in which one can cause an electron to tunnel between two closely spaced layers, [35] or to tunnel from a three-dimensional electron system into the center [36] or into the edge of a two-dimensional system. [37] In the quantum Hall regime, one encounters a variety of complex behaviors, where the tunneling current depends in a non-linear fashion on the voltage applied. [38] This behavior has been much studied, but is only partially understood in real experimental systems. Other experiments, which study deviations from the quantized Hall conductance due to tunneling of quasiparticles across a narrow constric-

tion (referred to earlier) also show non-linear dependences on the applied voltage, which have been much studied and are only partially understood. [39] Experiments that measure shot noise in this tunneling regime have led to a direct measure of the fractional charge of quasiparticles, but are also only partially understood. [40]

Acknowledgments

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